(A) Uniqueness of the minimizer

(B) When the minimizer is a weak solution of E-L

(C) Polyconvex case - elasticity - L has det

(D) When a weak solution is a minimizer

We shall give some answers to these questions/topics

1st: The existence result holds for $L(P, u, x)$ $\mathbb{R}^{m\times n} \times \mathbb{R}^m \times U$ case

Polyconvexity allows the lower-semi-continuous results when the convexity on P does not hold.

(details to be explained)

Polyconvex. $L(P, \underbrace{\det P}_{r}, u, x)$ is convex in $\underbrace{P \,\&\, r}$

E.g. $L = (\det P) = L(P, r, u, x) = r$ — Not convex as a function of $P = (P^i_j)$

But it is clearly convex in r

$$\begin{pmatrix} u^1_1 & u^2_1 \\ u^1_2 & u^2_2 \end{pmatrix} - \begin{pmatrix} p^1_1 & p^2_1 \\ p^1_2 & p^2_2 \end{pmatrix} \quad - \det(\square) = \underset{z^1}{p^1_1}\underset{z^3}{p^2_2} - \underset{z^2}{p^1_2}\underset{z^4}{p^2_1}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Morrey: $u: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$L(P, r, u, x) = \sqrt{1 + (p^1_1)^2 + (p^1_2)^2 + (p^2_1)^2 + (p^2_2)^2 + \boxed{\det\underbrace{\begin{pmatrix} p^1_1 & p^2_1 \\ p^1_2 & p^2_2 \end{pmatrix}^2}_{r^2}}}$$

Check this is polyconvex!

2nd Some variational problem involve the higher derivatives

e.g a. Thin plate

$$L = \alpha \underbrace{(u_{xx} + u_{yy})}_{\Delta u}^2 + \beta (u_{xx}u_{yy} - u_{xy}^2)$$

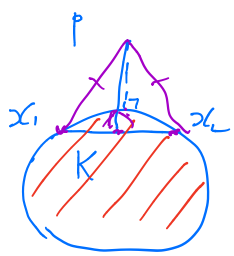
E-L equation is 4-th order

b. Hilbert action in GR (general relativity) also has a Lagrangian of 2nd-derivatives. Miraculously E-L still is 2nd order i.e. Einstein equation

Now we start our A.B.C.D.

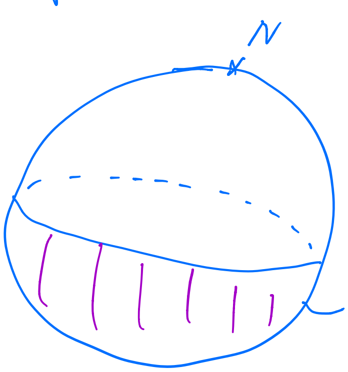
(A) Uniqueness: closest point $q \in K$ is unique, K is convex

E.g



$\Longrightarrow$ q is unique

If $\exists$ two x_1, x_2 $\Longrightarrow$ $\boxed{d\left(p, \frac{x_1+x_2}{2}\right) < \max\left(d(p, x_i)\right)}$ by triangle geometry.



— the minimizer is NOT unique

S^2_- — closed & convex in S^2

We need some convexity on the functional & convexity on the set.

<u>Theorem</u>. $L(p, x)$ satisfies $L_{p_i p_j}\xi^i\xi^j \geq \theta|\xi|^2$

Then $I(u)$ has only 1 minimizer in $\mathcal{A}$ - convex

(e.g. $W^{1,q}(\Omega)$, $1<q<\infty$ if $L(p,x) \geq \alpha|p|^q$)

<u>proof</u>. Same idea. Let $v = \frac{u_1+u_2}{2} \in \mathcal{A}$

$I(u_i) = \inf_{u\in\mathcal{A}} I(u)$.

$$\boxed{\mathcal{A} = \{w \in W^{1,q}(\Omega),\ w|_{\partial\Omega} = f\} \text{ is convex}}$$

Now we use the Taylor expansion & assumption

$$\boxed{L(q, x) = L(p, x) + L_p(p, x)\cdot(q-p) + \frac{1}{2} L_{p_i p_j}(p', x)(q_i - p_i)\cdot(q_j - p_j)}$$

$$\geq L(p, x) + L_p(p, x)\cdot(q-p) + \boxed{\frac{\theta}{2}}|q-p|^2$$

Apply it to $q = \nabla u_1$ $p = \frac{\nabla u_1 + \nabla u_2}{2} = \nabla v$

$$\Rightarrow \underbrace{I(u_1)}_{\int L(\nabla u_1, x)} \geq I(v) + \int_U L_p(\nabla v)\cdot\left(\frac{\nabla u_1 - \nabla u_2}{2}\right)$$

$$+ \frac{\theta}{2}\int_U \left|\frac{\nabla u_1 - \nabla u_2}{2}\right|^2$$

Similarly, $\underset{q = \nabla u_2}{I(u_2)} \geq I(v) + \int_U L_p(\nabla v)\cdot\left(\frac{\nabla u_2 - \nabla u_1}{2}\right) + \frac{\theta}{2}\int \square$

$$\Rightarrow \quad \boxed{\frac{I(u_1)+I(u_2)}{2} \geq I(v) + \frac{\theta}{4}\int_U |Du_1 - Du_2|^2} \geq \frac{I(u_1)+I(u_2)}{2}$$

But $\quad I(v) \geq I(u_i) = m$

$$\Rightarrow \quad Du_1 - Du_2 = 0 \quad \text{a.e}$$

$$\Leftrightarrow \quad u_1 - u_2 = c \quad \text{a.e}$$

$$\Rightarrow \quad c = 0 \quad \text{by } u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = f.$$

(B) The minimizer satisfies the E-L weakly.

Weak solution in $W^{1,q}$ — sense

More structure conditions: If in distribution sense much weaker conditions are needed.

$$\begin{cases} |L(p,z,x)| \leq A(|p|^q + |z|^q + 1) \\ |L_p| + |L_z| \leq A(|p|^{q-1} + |z|^{q-1} + 1) \end{cases}$$

$\forall \eta \in W_0^{1,q}$

$$\int_\Omega L_{p_i}(Dz, z, x) D_i\eta + L_z(Dz, z, x)\eta = 0$$

Theorem: the minimizer of $I(z) = \int_U L(Dz, z, x)$

Satisfi $\boxed{-(L_{p_i})_{x_i} + L_z = 0}$

Proof: The key is to justify the calculation of the derivative of $I'(t)|_{t=0}$ — since now $u \in W^{1,q}(\Omega)$ only.

$$\eta \in W_0^{1,q}(\Omega)$$

$$I(t) = \int_\Omega L(\nabla(u+t\eta), u+t\eta, x)$$

$$\lim_{t\to 0} \boxed{\frac{I(t)-I(0)}{t}} = \lim_{t\to 0} \int_\Omega \frac{L(\nabla u + t\nabla\eta, u+t\eta, x) - L(\nabla u, u, x)}{t}$$

if exists $\to \overset{\parallel}{0}$

$$= \int_\Omega (L_p \cdot \nabla\eta + L_z \eta).$$

Clearly

$$\lim_{t\to 0} \underbrace{\frac{L(\nabla u + t\nabla\eta, u+t\eta, x) - L(\nabla u, u, x)}{t}}_{E_t}$$

$$= \underbrace{L_p \nabla\eta + L_z \eta}$$

pointwisely or a.e

since L is smooth

Write

$$E = \underbrace{\frac{1}{t}\int_0^t L_p(\nabla u + s\nabla\eta, u+s\eta, x) \overset{\nabla\eta}{\cdot} + \underline{L_z(\cdots)\eta}\, ds}_{\frac{d}{ds} L(\nabla u + s\nabla\eta, u+s\eta, x)}$$

$$\Rightarrow |E| \le A\Big(\underbrace{\underline{|\nabla u|^{q-1} + |\nabla\eta|^{q-1}}}_{\in L^{q'}} + \underline{|u|^{q-1} + |\eta|^{q-1}} + 1\Big)\underbrace{(|\nabla\eta| + |\eta|)}_{\in L^q}$$

$\frac{1}{q'} + \frac{1}{q} = 1$ $\quad q' = \frac{q}{q-1}$

Hence $\int_\Omega |E| < +\infty \quad \forall t$

DCT $\Rightarrow$ $\lim_{t\to 0}$ can be exchanged

$$\Rightarrow 0 = \int_U (L_p \nabla\eta + L_z \eta) \quad \forall \eta \in W^{1,q'}$$

(C) Lower Semi-Continuity under Poly-convexity.

This is for the case $u : U \to \mathbb{R}^n$

$$L(P, r, u, x) = L(u^i_{x_j}, \det(u^i_{x_j}), u, x)$$

As before if $L(P, r, u, x) \geq \boxed{\alpha |P|^q} - \beta$

the existence follows from the semi-continuity (lower) under the weak limit.

$$I(u) = \int L(Du, \det(Du), u, x) \geq \alpha \int |Du|^q - \beta' |\Omega|$$

Theorem: L is poly-convex $\overset{L \geq 0}{\Longrightarrow}$

$\forall\, n < q < \infty$, $\mathcal{A} = W^{1,q}(U, \mathbb{R}^n) \cap \{u|_{\partial U} = f\}$

Then $I(u) = \int_U L(Du, \det(Du), u, x)$

is lower-semi-continuous (weakly).

Pf $u_k \rightharpoonup u$ in $W^{1,q}$ $\Longrightarrow$ $\forall \varepsilon > 0$

$$\sup_k \left(\int_U |u_k|^q + |Du_k|^q\right) \leq A$$

pick E_ε, F_ε as before

$u_k \to u$ uniformly

$|Du| + |u| < \frac{1}{\varepsilon}$ on F_ε

want to show

$$\varliminf_{k \to \infty} I(u_k) \geq \int_{G_\varepsilon} L(Du, \det(Du), u, x)$$

$G_\varepsilon = E_\varepsilon \cap F_\varepsilon$

$$I(u_k) = \int_U L(\nabla u_k, \det(\nabla u_k), u_k, x) \geq \int_{G_\varepsilon} L(\nabla u_k, \det(\nabla u_k), u_k, x)\, dx$$

$$\geq \int_{G_\varepsilon} L(\nabla u, \det(\nabla u), u_k, x) \quad \mathrm{I} \quad \left(\to \int_{G_\varepsilon} L(\nabla u, \det(\nabla u), u, x) \right)$$

$$+ \int_{G_\varepsilon} L_p \cdot (\nabla u_k - \nabla u) + L_r \cdot (\det(\nabla u_k) - \det(\nabla u)) \quad \mathrm{II} \quad \mathrm{III}$$

I & II can be handled by the method before!

$\nabla u_k \to \nabla u$ in L^q hence we are in the same situation as before

$u_k \to u$ in $W^{1,q}$ $\Rightarrow$ $u_k \to u$ in L^q

$\nabla u_k \to \nabla u$ in L^q

For $L_p(\nabla u, \det(\nabla u), u_k, x)$ split into 2 terms

$$\mathrm{II} \to \int_{G_\varepsilon} L_p(\nabla u, \det(\nabla u), u, x) \cdot (\nabla u_k - \nabla u) \to 0$$

by weak convergence

$$\& \int_{G_\varepsilon} \left(L_p(\nabla u, \det(\nabla u), u_k, x) - L_p(\nabla u, \det(\nabla u), u, x) \right) \cdot (\nabla u_k - \nabla u) \to 0$$

by DCT.

The result follows if we can show

$$\mathrm{III} \quad \boxed{\det(\nabla u_k) \to \det(\nabla u)} \quad \text{in } L^{q/n}$$

$\nabla u_k \to \nabla u$

Proposition: $u_k \xrightarrow[W]{} u$ in $W^{1,\varepsilon}(U, \mathbb{R}^n)$

$$\Rightarrow \quad \det(\nabla u_k) \xrightarrow[w]{} \det(\nabla u) \quad \text{in } L^{q/n}$$

$q/n > 1$

This result can be useful in other situations.

Pf: Induction on matrix rank. Recall $\forall w : \Omega \to \mathbb{R}^n$ smooth

$$\det(Dw) = \sum w^i_{x_j} (\operatorname{cof} Dw)^i_j$$

$A = (a^i_j)$

$A^i_j = \operatorname{cof}(A)$

$$\Rightarrow \quad = \sum \left(w^i (\operatorname{cof} Dw)^i_j \right)_{x_j}$$

Here we used the result $\boxed{\sum_j \left((\operatorname{cof} Dw)^i_j \right)_{x_j} = 0}$

$\Rightarrow \forall \varphi \in C_c^\infty(\Omega)$

$$\int_\Omega \varphi \det(Dw) = -\int_\Omega \varphi_{x_j} w^i \operatorname{cof}(Dw)^i_j \qquad (*_1)$$

Now let $w \in W^{1,q}(\Omega)$, w_ε be the mollifier

$$\Rightarrow \int \varphi \det(Dw_\varepsilon) = -\int \varphi_{x_j} w_\varepsilon^i \operatorname{cof}(Dw_\varepsilon)^i_j$$

[EG] $\to$ $\downarrow$ $\varepsilon \to 0$

$$\int \varphi \det(Dw) = -\int \varphi_{x_j} w^i \operatorname{cof}(Dw)^i_j \qquad (*)$$

By DCT

Now we want to show if $u_k \to u$

$\alpha = q/n$

α'

$$\int \varphi \det(DU_k) \longrightarrow \int \varphi \det(Du)$$

$(\forall \varphi \in L^{\alpha'}$, but it is enough to establish it for $\varphi \in C_c^\infty(\Omega)$. Since $\int |DU_k|^q + \int |U_k|^q \le A$

dual $\alpha' \to \alpha = q/n$

$\int |\det(DU_k)|^\alpha \le A'$ & approximation will extends to $\varphi \in L^{\alpha'}$

$\Longrightarrow$

$$\int |a^1_{i_1} \cdots a^n_{i_n}|^\alpha = \int |a^1_{i_1}|^\alpha \cdots |a^n_{i_n}|^\alpha$$
$$\le \Big[\int |a^1_{i_1}|^{\underbrace{\alpha n}_{q}}\Big]^{\frac{\alpha}{n}} \cdots \Big[\int |a^n_{i_n}|^{\alpha n}\Big]^{\frac{\alpha}{n}}$$

Now we apply $(*1)$

$\psi \sim \psi_k$ need Morrey's Lemma

$$\int \varphi \det(U_k) = -\int \boxed{\cancel{\varphi_{x_j} (U_k)^i}} (\operatorname{cof} DU_k)^i_j$$

$$\int \underbrace{(\operatorname{cof}(DU_k))}^i_j \psi \longrightarrow \int \underbrace{(\operatorname{cof}(Du))}^i_j \psi$$

$\uparrow$ By induction

$\forall \psi \in C_c^\infty(\Omega)$.

$$\Longrightarrow \int \varphi_{x_j} u^i \operatorname{cof}(DU_k)^i_j \longrightarrow \int \varphi_{x_j} u^i \operatorname{cof}(Du)^i_j$$

$\in L^q$

Then we only need to check

$$\int \varphi_{x_j} \underbrace{(u^i_k - u^i)} \operatorname{cof}(DU_k)^i_j \to 0$$

$\rightsquigarrow$ This can be done by similar arguement as before $\int |\varphi_{x_j} \operatorname{cof}(DU_k)^i_j| \le A$

(D) When a critical point is a minimizer?

We only look at very special cases

[$\exists$ Weierstrass' necessary condition which we shall not get into
See Morrey's book if interested]

<u>Proposition</u>: If $L(p, z, x)$ is convex in p, z

$\Rightarrow$ Any critical point is a minimizer.

$= W^{1,2}(\Omega) \cap \{u|_{\partial\Omega} = f\}$

<u>Pf</u>: $\forall\, w \in \mathcal{A}$ & u a solution to E-L equation

$\eta = w - u$

$$L(Dw, w, x) - L(Du, u, x)$$

$$\geq L_p(\underline{\underline{Du, u}}, x)\cdot(\underbrace{Dw - Du}_{D\eta}) + L_z(Du, u, x)\,(\underbrace{w-u}_{\eta})$$

$\left\{ \begin{array}{l} w, u \in \mathcal{A} \\ u \text{ solves} \\ \text{the E-L} \\ \text{weakl} \end{array} \right.$

$$\Rightarrow I(w) - I(u) \geq \int \underbrace{\partial_{x_j}\eta \cdot \left(L_p(Du, u, x)\right) + L_z(Du, u, x)\cdot\eta}_{=0}$$

Otherwise RHS is understood as a PDE weakly on $\mathcal{A}$, namely weakly in $W^{1,2}(\Omega)$ with $w|_{\partial\Omega} = f$ (hence $w - u|_{\partial\Omega} = 0$)

$\eta \in W_0^{1,2}(\Omega)$

If only assume that $L(p, z, x)$ convex in p, the problem is NOT easy

$n = 1$ case $\exists$ theory of Jacobi (See Gelfand-Fomin Ch5)

"No conjugate point $\Rightarrow$ the critical point is a local minimal"

· The more general case $L(p, z, x)$ is only convex in p.

Here we assume u is a smooth solution to the Euler-Lagrange equation

$$\left(-L_{p_i}(\nabla u, u, x)\right)_{x_i} + L_z(\nabla u, u, x) = 0 \qquad \text{(E-L) of}$$

Assume that $u(x, \lambda)$ is a family of solutions with $\lambda \in I$ — an interval containing 0 & $u(x, 0) = u(x)$.

Assume $\theta : \Omega \to I$ satisfies $\theta|_{\partial\Omega} = 0$

Let $\mathcal{R} = \left\{ w \;\middle|\; w(x) = u(x, \theta(x)),\ \forall \theta \right\}$

$$w(x)\Big|_{\partial\Omega} = u(x, 0)\Big|_{\partial\Omega} = u(x)\Big|_{\partial\Omega}$$

Theorem: $I(u) \le I(w) \quad \forall w \in \mathcal{R}$, provided L is convex in p.

(Theorem 10 of §8.2).

Review §6.3 of Evans on the regularity of linear PDEs.

$$\left|\int (f_k - f)\varphi\right| \le \underbrace{\left|\int (f_k - f)\psi\right|}_{\to 0} + \left|\int (f_k - f)(\varphi - \psi)\right| \le \underbrace{\|f_k - f\|_{L^\alpha}}_{\le A} \underbrace{\|\varphi - \psi\|_{L^{\alpha'}}}_{\text{arbitrarily small}} \quad \forall \varphi \in L^{\alpha'},\ \psi \in C_c^\infty(\Omega).$$

$$\int \left|\varphi f_\varepsilon^1 \cdots f_\varepsilon^n - \varphi f^1 \cdots f^n\right|$$

$$= \int |\varphi (f_\varepsilon^1 - f^1) f_\varepsilon^2 \cdots f_\varepsilon^n| + \cdots \quad |\varphi f^1 \cdots f^{n-1}(f_\varepsilon^n - f^n)|$$

$$\le \|f_\varepsilon^1 - f\|_{L^q} A + \cdots + \|f_\varepsilon^n - f^n\|_{L^q} A \to 0 \text{ as } \varepsilon \to 0$$